STABILITY OF A SPHERICAL STAR SYSTEM

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UDC 523.11

The stability of a spherically symmetric aggregate of point gravitating particles relative to arbitrary small perturbations is studied. It is assumed that in the absence of perturbations the particles move along circular trajectories chaotically oriented in space so that the total moment of the aggregate is zero. Dimensions of the aggregate are large in comparison to the gravitational radius, and particle velocities are nonrelativistic. It is shown that there exist initial mass-density distributions unstable relative to any perturbations with the exception of radial and dipole perturbations. A general stability criterion is formulated, with the form $d\Omega^2/dr > 0$, where $\Omega^2 = (4\pi G / r^3)\int\limits_0^r \rho_0 r^2 dr$, $\rho_0(r)$ is the aggregate mass density, and G is the gravitational constant. The dependence of the increment on l, the perturbation harmonic number, is studied. In the case of weak inhomogeneity $r(d\Omega^2/dr)/\Omega^2 \ll 1$ the increment is maximum for quadrupole perturbations ($l = 2$) and decreases monotonically with increase in l. In the opposite case of high inhomogeneity $r(d\Omega^2/dr)/\Omega^2 \gg l^2$ the increment increases with increase in l. In the case of weak inhomogeneity the increment may be as small as desired. For high inhomogeneity, instability develops over a time period smaller than the period of revolution of an individual particle. For $d\Omega^2/dr < 0$ the system is stable. Consideration of system microstructure in this case leads to damping of macrooscillations (system "heating").

1. The trajectory of every particle is characterized by an angular velocity vector Ω

$$\Omega^2(r) = \frac{1}{r}\frac{d\Phi_0}{dr} = 4\pi G r^{-3}\int\limits_0^r \rho_0(r)\, r^2\, dr \qquad (1.1)$$

Here Φ_0 is the self-congruent gravitational potential satisfying the equation

$$\Delta\Phi_0(r) = 4\pi G\rho_0(r) \qquad (1.2)$$

The dimensions of the aggregate R are assumed much greater than its gravitational radius $r_g = 2MG/c^2$, where M is the mass of the aggregate; c is the speed of light ($R \gtrsim 10r_g$), i.e., the system obeys the laws of classical mechanics. No assumptions are made relative to the form of the density $\rho_0(r)$.

Let the system undergo a small arbitrary perturbation. In [1], using the self-congruent field approximation an equation was obtained describing the natural system oscillations developing in such a case. The perturbed self-congruent potential was represented in the form of a superposition of spherical harmonics. In view of the linearity of the equations, the system was solved for an individual harmonic of the perturbed potential $\Phi_1(r, \theta, \varphi, t) = \chi_l(r, \omega) Y_{lm}(\theta, \varphi) e^{-i\omega t}$. For the radical portion of each harmonic $\chi_l(r, \omega)$ an equation was obtained,

$$A_l(r, \omega)\frac{\partial^2\chi_l(r, \omega)}{\partial r^2} + \left(\frac{\partial A_l(r, \omega)}{\partial r} + \frac{2A_l(r, \omega)}{r}\right)\frac{\partial \chi_l(r, \omega)}{\partial r} - \frac{B_l(r, \omega)}{r}\chi_l(r, \omega) = 0 \qquad (1.3)$$

Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, pp. 63-73, November-December, 1974. Original article submitted May 4, 1973.

Here $\chi_l(r, \omega)$ is the radial portion of the l-th harmonic of the perturbed self-congruent potential in Fourier representation

$$A_l(r, \omega) = 1 + \Omega_g^2 \sum_{s=-l}^{l} |P_{s0}(\pi/2)|^2 / (\omega_s^2 - \Omega^2 - \Omega_g^2) \tag{1.4}$$

$$B_l(r, \omega) = r^2 \frac{d}{dr}\left\{\frac{\Omega_y^2}{r} \sum_{s=-l}^{l} |P_{s0}(\pi/2)|^2 \frac{2s\Omega}{\omega_s(\omega_s^2 - \Omega^2 - \Omega_g^2)}\right\} +$$
$$+ \Omega_g^2 \sum_{s=-l}^{l} |P_{s0}{}^l(\pi/2)|^2 \left\{\frac{4s\Omega\omega_s + s^2(\omega_s^2 - \Omega_g^2 + 3\Omega^2)}{\omega_s^2(\omega_s^2 - \Omega^2 - \Omega_g^2)} + \frac{\alpha_{s+1}'^2}{\omega_s(\omega_s - 2\Omega)}\right\} + l(l+1) \tag{1.5}$$

where $P^l{}_{mn}(\theta)$ are generalized Legendre polynomials [2], in particular,

$$P_{s0}{}^l(\pi/2) = \begin{cases} \dfrac{(l+s)!\,(l-s)!}{\left[2^l\left(\frac{l+s}{2}\right)!\left(\frac{l-s}{2}\right)!\right]^2}, & \text{if} \quad (l+s) \text{ is even} \\ 0, & \text{if} \quad (l+s) \text{ is odd} \end{cases} \tag{1.6}$$

$$\sum_{s=-l}^{l} |P_{s0}^l(\pi/2)|^2 = 1 \tag{1.7}$$

$$\omega_s = \omega - s\Omega, \quad \alpha_s' = \sqrt{(l+s)(l-s+1)} \tag{1.8}$$
$$\Omega_g^2(r) = 3\Omega^2(r) + r\frac{d\Omega^2}{dr} = 4\pi G\rho_0$$

and $\rho_0 = \rho_0(r)$ is the perturbed mass density of the system.

In the present study the frequency spectrum of natural oscillations obtained from Eq. (1.3) will be studied. We will show that among the eigenfrequencies of the system there may be imaginary values if the condition $d\Omega^2/dr > 0$ is fulfilled. This criterion indicates that in the case of monotonic increase of the initial mass density to the edge of the aggregate the system is known beforehand to be unstable. The dependence of the instability increment on l, the perturbation harmonic number, will be studied. In the case of weak inhomogeneity $r(d\Omega^2/dr) \ll \Omega^2$ the increment is maximized at $l = 2$ and decreases monotonically with increase in l.

The stability criterion (Sec. 5) has the form $d\Omega^2/dr < 0$. Hence, it follows that if the density decreases monotonically toward the edge of the aggregate, then the system is unconditionally stable. We will also show that at $d\Omega^2/dr < 0$, the natural system oscillations decay with time. These results are valid only for perturbations under the influence of which individual particles are not displaced by a distance of the order of the mean interparticle distance during the course of a revolution about the center of the aggregate. In the opposite case, linear instability occurs (the density amplitude increases linearly with time).

2. We will consider the solution of the system oscillation equation. In [1] an expression was obtained for Fourier harmonics of the macroscopic material velocity of the perturbed system, defined as $W(r) = j(r)/\rho(r)$, where j is the material flux density. The expression for the spherical component $W_r(r)$ of the velocity $W(r)$ has the form

$$W_r(r, \theta, \varphi, t) = V_0(r, \omega)\, T^l_{m,0}(\pi/2 - \varphi, \theta, 0) \tag{2.1}$$

The coefficient $V_0(r, \omega)$ is determined by the radial portion of the perturbed self-congruent potential $\chi_l(r, \omega)$ through the formula

$$V_0(r, \omega) = -i \sum_{s=-l}^{l} \frac{\omega}{\omega_s^2 - \Omega^2 - \Omega_g^2} \mathcal{P}_{l,s}(\pi/2)$$
$$\times T_{0s}{}^l(0, -\pi/2, -\pi/2)\left\{\frac{\partial\chi_l(r, \omega)}{\partial r} - \frac{2s\Omega(r)}{\omega_s}\frac{\chi_l(r, \omega)}{r}\right\} \tag{2.2}$$

where $\mathcal{P}_{ls}(\pi/2)$ is the unified Legendre polynomial; ω_s and Ω_g^2 are defined by Eq. (1.8). From the definition of $\mathcal{P}_{ls}(\theta)$ we have $\mathcal{P}_s(\pi/2) = 0$ if $l + s$ is odd; thus, the index s in the sum (2.2) takes on values $s = -l, -l + 2, \ldots, l - 2, l$.

The total macroscopic velocity corresponding to arbitrary initial conditions is written in the form of the Fourier integral of Eq. (2.1),

$$W(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(r, \omega) e^{-i\omega t} d\omega \tag{2.3}$$

We will assume that system perturbations develop at a moment in time $t_0 = 0$, so that

$$W(r, t) = 0 \quad \text{where } t < 0 \tag{2.4}$$

We will examine Eq. (1.3) for the function $\chi_l(r, \omega)$. This equation is soluble [3]. It is sufficient to know the behavior of its solution near the singular points, which permits calculation of integral (2.3). The singular points in the given case are the poles of the coefficients $B_l(r, \omega)$ and $A_l(r, \omega)$ and the zeroes of $A_l(r, \omega)$. The function $B_l(r, \omega)$ has first- and second-order poles relative to r at points determined from the equations

$$\omega_s(r) \quad 0 \tag{2.5}$$

$$\omega^2_s - \Omega^2 - \Omega_g^2 = 0 \tag{2.6}$$

The remaining poles $B_l(r, \omega)$ in the notation of Eq. (1.5) disappear after reduction with use of the relationship

$$\sum_{s=-l}^{l} |P_{s0}{}^l(\pi/2)|^2 \frac{\alpha'^2_{s+1}}{\omega_s(\omega_s - 2\Omega)} = \sum_{s=-l}^{l} |P_{s0}{}^l(\pi/2)|^2 \frac{s}{\Omega\omega_s}$$

Near the first-order poles (2.5), solutions of Eq. (1.3) have the form

$$\chi_1 = (r - r_{B'}) \sum_{k=0}^{\infty} c_k'(r - r_{B'})^k$$

$$\chi_2 = C\chi_1 \ln(r - r_{B'}) - 1 + \sum_{k=1}^{\infty} d_k'(r - r_{B'})^k \tag{2.7}$$

where $r_{B'}$ are roots of Eq. (2.5), the numbers c_k' and d_k' are determined from Eq. (1.3), and $C = b'_{-1}/r_B'^2\alpha_0'$, b_{-1}' and α_0' of the Laurent expansion of the functions $B_l(r, \omega)$ and $A_l(r, \omega)$ near the point r_B'. Near the second-order poles (2.6), coinciding with the first-order poles of $A_l(r, \omega)$, we have

$$\chi_1 = (r - r_B)^2 \sum_{k=0}^{\infty} c_k (r - r_B)^k$$

$$\chi_2 = f\chi_1 \ln(r - r_B) - \frac{1}{2} + \sum_{k=1}^{\infty} d_k (r - r_B)^k \tag{2.8}$$

$$f = \frac{1}{2r_B^2 a_{-1}} \left[\frac{(b_{-2})^2}{2r_B a_{-1}} - \frac{b_{-2}}{r_B} - \frac{b_{-1}}{2} \right]$$

where r_B are roots of Eq. (2.7), the numbers d_k and c_k are determined from Eq. (1.3), and the coefficients b_{-2}, b_{-1} and α_{-1} are found from expansion of the functions $B_l(r, \omega)$ and $A_l(r, \omega)$ in a Laurent series near the point r_B. Near the zeroes of the coefficient $A_l(r, \omega)$ the solution has the form

$$\chi_1 = \sum_{k=0}^{\infty} c_k''(r - r_A)^k$$

$$\chi_2 = \chi_1 \ln(r - r_A) + \sum_{k=1}^{\infty} d_k''(r - r_A)^k \tag{2.9}$$

where r_A are the roots of the equation

$$A_l(r, \omega) = 1 + \Omega_g^2 \sum_{s=-l}^{l} \frac{|P_{s0}{}^l(\pi/2)|^2}{\omega_s^2 - \Omega^2 - \Omega_g^2} = 0 \tag{2.10}$$

This equation contains no odd powers of ω, i.e., $r_A = r_A(\omega^2)$. In the derivation of Eq. (2.8) it was assumed that the function $A_l(r, \omega)$ has nulls of only first order relative to r. For some values of r the appearance of second-order nulls is possible, which leads to damping of density oscillations at these points. The solutions of Eq. (2.7)-(2.9) are determined to within the accuracy of multiplication by an arbitrary function ω.

We will calculate the contribution of Eqs. (2.7)-(2.9) to the integral of Eq. (2.3). For this purpose we will consider the coefficient $V_0(r, \omega)$ appearing in Eq. (2.1). Substituting Eq. (2.7) in Eq. (2.2), we find that $V_0(r, \omega)$ contains a term with the form $\ln(r - r_B'(\omega))$. Thus, the function $V_0(r, \omega)$ with respect to r has a logarithmic singularity at the points $\omega = \omega_B' \equiv s\Omega$ [in light of Eq. (1.6), the index s takes on the values $s = -l, -l + 2, \ldots, l - 2, l$]. Substituting Eq. (2.8) in Eq. (2.2), we find that $V_0(r, \omega)$ also has singularities of the form $(\omega - \omega_B(r))^{-1}$ and $\ln(r - r_B(\omega))$ at the points $\omega = \omega_B(r)$, where $\omega_B(r)$ are roots of Eq. (2.6). The functions (2.9) produce a contribution to $V_0(r, \omega)$ of the form $\ln(r - r_A(\omega))$ and $(r - r_A(\omega))^{-1}$. We will now find the contribution of these singularities to integral (2.3). We will take sections along the real axis in the complex plane ω and close the contour by integration from below. Circumvention of the poles in accordance with Eq. (2.4) is performed such that all poles fall within the integration contour. Calculating the integral (2.3), we have

$$W_r(r, l) = \Big\{ \sum_{s=-l}^{l} g_s(r) \cos[\omega_B(r, s) t + \psi_s(r)] + \sum_A f_A(r) \cos[\omega_A(r) t + \varphi_A(r)] + F(r, t) \Big\} T_{m0}{}^{l}(\pi/2 - \varphi, \theta, 0) \tag{2.11}$$

Here $\omega_B(r, s) = s\Omega \pm \sqrt{\Omega^2 + \Omega_g^2}$, and the frequencies ω_A are found from Eq. (2.10). In Sec. 3 it will be shown that all frequencies $\omega_A(r)$ are different. Therefore, all poles producing a contribution to the second sum are of first order. In the first sum the index s takes on values $S = -l + 2, \ldots, l - 2, l$.

The amplitudes $g_s(r)$, $f_A(r)$ and phases $\psi_s(r)$, $\varphi_A(r)$ may be calculated if the complete solution of Eq. (1.3) is known. The function $F(r, t)$ is the integral over dimensions of the terms containing logarithms. The concrete form of this function, just as the form of the amplitudes and phases, is not of interest. For large times $F(r, t)$ decreases as t^{-1}.

The frequencies $\omega_B(r, s)$ are always real. The frequencies $\omega_A(r)$ may be real or imaginary, depending on the form of the initial density $\rho_0(r)$. In Sec. 3 conditions will be formulated for $\rho_0(r)$, under which ω_A are imaginary.

3. We will consider unstable distributions of initial density. We introduce the notation

$$X = \omega^2 / \Omega^2(r), \quad \mu^2 = \Omega_g^2 / \Omega^2 + 1$$
$$\alpha_s = \mu + s, \quad \beta_s = \mu - s \tag{3.1}$$

The function $A_l(r, \omega)$ may be written in this notation in the form

$$A_l(r, \omega) = 1 + 2(\mu^2 - 1) \sum_{s=1}^{l} |P_{s0}(\pi/2)|^2 \frac{X - \alpha_s\beta_s}{(X - \alpha_s)^2 (X - \beta_s)^2} + (\mu^2 - 1) \frac{|P_{00}{}^{l}(\pi/2)|^2}{X - \mu^2} \tag{3.2}$$

For each given l we will examine such initial density distributions $\rho_0(r)$ for which

$$\mu^2 = \Omega_g^2 / \Omega^2 + 1 > l^2 \quad (l > 2) \tag{3.3}$$

We consider the equation

$$1 + 2(\mu^2 - 1) \sum_{s=1}^{l} |P_{s0}{}^{l}(\pi/2)|^2 \frac{X - \alpha_s\beta_s}{(X - \alpha_s)^2 (X - \beta_s)^2} + (\mu^2 - 1) \frac{|P_{00}{}^{l}(\pi/2)|^2}{X - \mu^2} = 0 \tag{3.4}$$

The squares of the frequencies $\omega_A{}^2$ are roots of this equation.

We will find the position of the frequencies ω_A in the complex plane ω. In light of Eq. (1.6), it follows that the cases of even and odd l must be considered separately. Let l be odd. Then

$$A_l(r, X) = 1 + 2(\mu^2 - 1) \sum_{s=1}^{l}{}' |P_s{}^{l}(\pi/2)|^2 \frac{X - \alpha_s\beta_s}{(X - \alpha_s)^2 (X - \beta_s)^2} \tag{3.5}$$

where Σ' denotes that summation is performed over odd s. This sum contains $l + {}^1/_2$ terms. Consequently, Eq. (3.4) has $(l + 1)$ roots. The function $A_l(r, X)$ relative to X has $(l + 1)$ positive first-order poles. We will show that $A_l(r, X)$ is monotonic in X. To do this, l real positive roots of Eq. (3.4) will be calculated, each of which lies between two neighboring positive poles of the function $A_l(r, X)$ (Fig. 1). We will calculate the derivative $\partial A_l/\partial X$

$$\frac{\partial A_l}{dX} = -2(\mu^2 - 1)\sum_{s=1}^{l}{}' |P_{s0}^l(\pi/2)|^2 \frac{X^2 - 2\alpha_s\beta_s X - \alpha_s^2\beta_s^2 + \alpha_s\beta_s(\alpha_s^2 + \beta_s^2)}{(X - \alpha_s^2)^2 (X - \beta_s^2)^2} \tag{3.6}$$

We will consider the quadratic trinomial $Q_s(X) = X^2 - 2\alpha_s\beta_s X - \alpha_s^2\lambda_s^2 + \alpha_s\beta_s(\alpha_s^2 + \beta_s^2)$, appearing in the numerator of every term of sum (3.6). Its roots $X = \mu^2 - s^2 \pm 2s\sqrt{-(\mu^2 - s^2)}$ are complex, if the condition of Eq. (4.3) is fulfilled. Consequently, $Q_s(x) > 0$ for all s and $\partial A_l/\partial X < 0$, i.e., the function $A_l(r, X)$ decreases monotonically. Thus, the l roots of the function $A_l(r, X)$ are real and positive. We will consider the remaining $(l + 1)$-th root. This root is real, since on the interval from $x = -\infty$ to the smallest pole $X = \beta_l^2 = (\mu - l)^2$ the function $A_l(r, X)$ decreases monotonically from 1 to $-\infty$, i.e., intersects the real axis. We will now determine the sign of this root. We calculate the value of $A_l(r, X)$ at $X = 0$.

$$\begin{aligned} A_l(r, 0) &= 1 - 2(\mu^2 - 1)\sum_{s=1}^{l}{}' \frac{|P_{s0}^l(\pi/2)|^2}{\alpha_s\beta_s} \\ &= 1 - 2\sum_{s=1}^{l}{}' |P_{s0}^l(\pi/2)|^2 \frac{\mu^2 - s^2 + s^2 - 1}{\mu^2 - s^2} = -\sum_{s=-l}^{l}{}' \frac{s^2 - 1}{\mu^2 - s^2} |P_{s0}^l(\pi/2)|^2 \end{aligned} \tag{3.7}$$

In deriving Eq. (3.7), Eq. (1.7) was utilized. Since $s^2 \geq 1$, $\mu^2 > l^2$, we have $A_l(r, 0) < 0$. Consequently, the root considered is negative (Fig. 1).

The case of l is treated analogously. The function $A_l(r, X)$ then takes on the form

$$A_l(r, X) = 1 + \frac{|P_{00}^l(\pi/2)|^2}{X - \mu^2}(\mu^2 - 1) + 2(\mu^2 - 1)\sum_{s=2}^{l}{}'' |P_{s0}^l(\pi/2)|^2 \frac{X - \alpha_s\beta_s}{(X - \alpha_s^2)(X - \beta_s^2)} \tag{3.8}$$

where Σ'' denotes summation over even s. As in the case of odd l the function $A_l(r, X)$ has $(l + 1)$ roots, of which l roots are positive [if Eq. (3.3) is satisfied]. To calculate the sign of the $(l + 1)$-th root we evaluate $A_l(r, 0)$

$$\begin{aligned} A_l(r, 0) &= 1 - |P_{00}^l(\pi/2)|^2 \frac{\mu^2 - 1}{\mu^2} \\ &- 2(\mu^2 - 1)\sum_{s=2}^{l}{}'' |P_{s0}^l(\pi/2)|^2 \frac{1}{\mu^2 - s^2} = -\sum_{s=-l}^{l}{}'' \frac{s^2 - 1}{\mu^2 - s^2} |P_{s0}^l(\pi/2)|^2 < \\ &- \sum_{s=2}^{l}{}'' \frac{s^2 - 1}{\mu^2} |P_{s0}^l(\pi/2)|^2 = \frac{1}{\mu^2}\Big(1 - \sum_{s=-l}^{l}{}'' s^2 |P_{s0}^l(\pi/2)|^2\Big) < 0 \end{aligned}$$

i.e., the root considered is negative.

Thus, it has been shown that the function $A_l(r, X)$ has one negative and l positive roots. It also follows from the proof that $A_l(r, X)$ has only first-order roots. In doing this initial density distributions $\rho_0(r)$ were considered for which $1 + \Omega_g^2/\Omega^2 > l^2$. We denote positive roots of Eq. (3.2) by $X_i = \omega_i^2/\Omega^2$ $(i = 1, 2, \ldots, l)$ and the negative root by $X_0 = -\omega_0^2/\Omega^2$. We may now write Eq. (2.11) in the form $(l > 2)$.

$$\begin{aligned} W_r = \Big\{ &\sum_{s=-l}^{l} g_s(r)\cos[\omega_B(r, s)t + \psi_s(r)] + \sum_{i=1}^{l} f_i(r)\cos[\omega_i(r)t + \varphi_i(r)] + H(r, t) + \\ &+ (f_+(r) + h_+(r, t))e^{\omega_0(r)t} + (f_-(r) + h_-(r, t))e^{-\omega_0(r)t}\Big\} T_{m0}^l(\pi/2 - \varphi, \theta, 0) \end{aligned} \tag{3.9}$$

Here the functions $h_\pm(r, t)$ may be represented as integrals over dimensions, performed from the points $\omega = \pm i\omega_0$. For large times the functions $H(r, t)$ and $h_\pm(r, t)$ decrease as t^{-1}.

The aggregate is exponentially unstable relative to perturbations containing lower harmonics (with the exception $l = 0.1$), if the density $\rho_0(r)$ satisfies Eq. (3.3). Relative to perturbations for $l = 0.1$ the system is stable for any initial distribution. In the case of radial perturbations $(l = 0)$ Eq. (3.2) has a single positive root $X_0 = 1$, or $\omega_0 = \pm\Omega(r)$. In the case of dipole perturbations $(l = 1)$ the system is also stable $(\omega_0 = 0)$.

These perturbations displace the center of gravity and will not be considered further. The following harmonic $l = 2$ leads to oscillations in the system at some mass distributions $\rho_0(r)$. Any real perturbation not possessing central symmetry possesses this harmonic.

We will consider initial distributions $\rho_0(r)$ satisfying the condition $\mu^2 > 4$. In view of the concepts discussed above these distributions are unstable with respect to any perturbations (with the exception of radial ones). Here $\mu^2 = 1 + \Omega_g^2/\Omega^2 = 4 + r(d\Omega^2/dr)/\Omega^2$. Thus, we have the instability criterion

$$d\Omega^2 / dr > 0 \tag{3.10}$$

where $\Omega^2(r)$ is determined from Eq. (1.1).

Distributions satisfying this criterion may also be unstable relative to higher perturbation harmonics ($l > \mu$). For example, at $\mu^2 = 4 + \varepsilon(r)$, where $0 < \varepsilon(r) \ll 1$, instability is produced by perturbations with even $l \geq 4$, and in the case $\mu^2 = 9 + \varepsilon(r)$ also by perturbations with odd $l \geq 5$. The proof is analogous to that given.

The dependence of the instability increment $\omega_0(r, l)$ on l is of interest. A study of this dependence requires cumbersome calculations. On the basis of what has been offered in Sec. 3 it may be assumed that the increment $\omega_0(l)$ decreases with increase in l. This, however, is invalid, since at $\mu^2 \gg l^2 > 1$ we have in the first nondisappearing order of $1/\mu^2$

$$\omega_0^2(r, l) = \Omega^2(r)\Big[\sum_{s=-l}^{l} s^2 |P_{s0}^l(\pi/2)|^2 - 1\Big] \tag{3.11}$$

Using Eqs. (1.6), (1.7), it is possible to prove that $\omega_0^2(l) = \omega_0^2(l-2) + 2l - 1$, $\omega_0^2(l) > \omega_0^2(l-1)$, i.e., the increment increases with increasing l. The other case is possible: for $\mu^2 = 9 + \varepsilon(r)$, where $0 < \varepsilon(r) \ll 1$, for the first two perturbation harmonics leading to instability from Eq. (3.3) we have

$$\omega_0^2(l=2) \sim 0.6\Omega^2(r), \quad \omega_0^2(l=3) \sim 0.3\varepsilon(r)\,\Omega^2 \tag{3.12}$$

i.e., the increment decreases. [For odd $l \geq 5$ it is possible to show that $\omega_0^2(l) \approx f(l)\varepsilon(r)\,\Omega^2$, where $f(l)$ is a decreasing function.] Instability with the increments of Eqs. (3.2), (3.3) develops rapidly — over a time of lower order than the period of revolution of the stars about the center of the aggregate ($\tau \lesssim 1/\Omega(r)$). Long-lived aggregates may be determined. For $\mu^2 = 4 + \varepsilon(r)$, where $0 < \varepsilon(r) \ll 1$ (this indicates a slow increase in density towards the edge of the system), for even perturbation harmonics producing instability, we have

$$\omega_0^2(l) = \gamma(l)\,\varepsilon(r)\Omega^2(r)$$

$$\gamma(l) = 3|P_{20}^l(\pi/2)|^2\Big[13|P_{20}^l(\pi/2)|^2 + 2|P_{00}^l(\pi/2)|^2 + \Big|16\sum_{s=4}^{l}\frac{s^2-1}{s^2-4}|P_{s0}^l(\pi/2)|^2\Big]^{-1} \tag{3.13}$$

decaying with increase in l. The maximum increment is $\omega_0^2(l = 2) \approx 0.2\varepsilon(r) \times \Omega^2$ and may be as small as desired.

We will consider the case of constant density $\rho_0(r) = \text{const}$. Stability of such a system was considered in [4, 5]. In those studies a conclusion was reached as to stability relative to arbitrary perturbations. In doing this the zero frequencies, whose existence follows from Eq. (3.4), were lost upon transition to the limit $\varepsilon \to 0$ ($\mu^2 = 4$ at $\rho_0 = \text{const}$). If $\varepsilon(r) < 0$, which indicates a slow decrease in density toward the system edge, the system is stable.

4. We will now consider stable density distributions. In Sec. 3 it was shown that for $\mu^2 > 4$ there is always an unstable perturbation mode ($l = 2$). We will consider distributions $\rho_0(r)$, corresponding to the inequality

$$1 < \mu^2 < 4 \tag{4.1}$$

Using the methods of Sec. 3, after cumbersome analysis we find (details are given in [6]) that in the case of odd l the function $A_l(X)$ has the form depicted in Fig. 2. For even l the function $A_l(X)$ is depicted in Fig. 3. Thus, under the conditions of Eq. (4.1) all nulls of the function $A_l(X)$ are positive, from which follows the stability of the given sys-

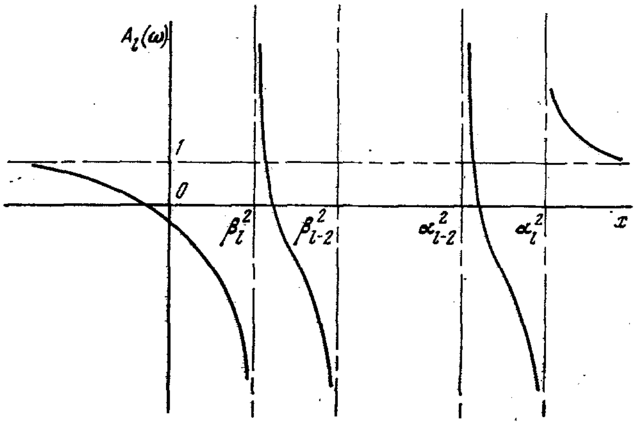

Fig. 1

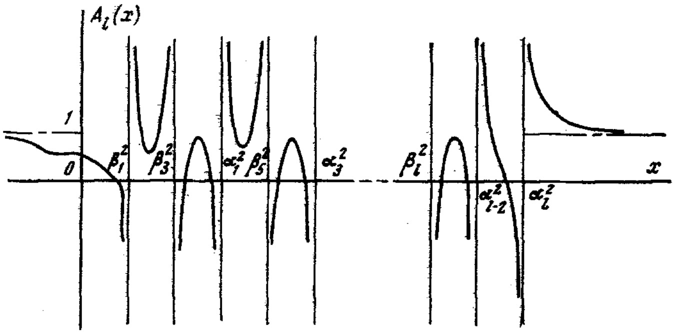

Fig. 2

tem. Using Eqs. (3.3) and (4.1) we find that the stability criterion has the form

$$d\Omega^2 / dr < 0 \tag{4.2}$$

We will now show that consideration of system microstructure leads to damping of oscillations. We introduce the characteristic dimension Δ, at which the system structure exerts an effect. This characteristic length in any case is greater than the distance between stars $r_0 \sim R/N^{1/3}$, where R is the system radius, and N is the number of stars.

We will consider the case of radial perturbations ($l = 0$). Let a perturbation at moment $t_0 = 0$ have the form

$$W_r(r,\ 0) = V(r), \quad W_\theta = W_\varphi = 0, \quad \rho_1(r,\ 0) = 0 \tag{4.3}$$

where V(r) is a small arbitrary function satisfying the inequality

$$V(r) \ll \Omega(r)\,\Delta \tag{4.4}$$

This condition means that the time necessary for a perturbed star to displace in radius by a distance ~Δ must be much greater than the period of revolution about the center of the system $T = 2\pi/\Omega(r)$. For a velocity $W(\mathbf{r},\ t)$ with initial conditions (3.1) we have

$$W_r(r,\ t) = V(r)\cos\Omega(r)\,t, \quad W_\theta = W_\varphi = 0 \tag{4.5}$$

Calculating the perturbed density ρ_1 from the Poisson equation we find

$$\rho_1(r,t) = -\frac{1}{\Omega}\frac{1}{r^2}\frac{d}{dr}(r^2\rho_0 V)\sin\Omega(r)\,t - \frac{\rho_0 V}{\Omega}\frac{d\Omega}{dr}\,t\cos\Omega(r)\,t \tag{4.6}$$

From Eq. (4.6) it follows that the amplitude of density oscillations increases linearly with time, and for $t_1 \sim \Omega(V(d\Omega/dr))^{-1}$ the linear approximation is inapplicable. For perturbations satisfying the condition (4.4), this is not the case. The frequency of velocity oscillation at two points at a distance of ~Δ differs by an amount ~Δ(dΩ/dr). Consequently,

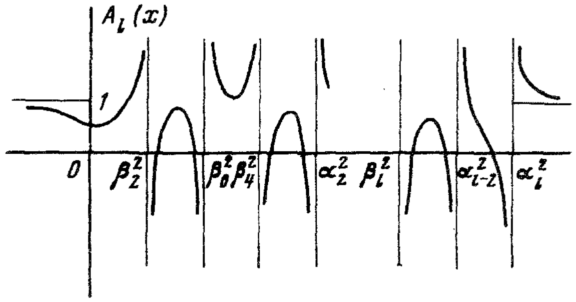

Fig. 3

after passage of a time $t \sim (\Delta(d\Omega/dr))^{-1}$ the velocities at these closely neighboring points will have opposite directions. Thus, a peculiar "dephasing" develops.

In [1] equations were derived for the macroscopic portions of the velocity **W** and density ρ. In doing this it was not considered that the role of a physical point must be played by a volume with dimensions $\sim\Delta$. Formation of opposing flows within a volume $\sim\Delta$ invalidates the condition dividing the macroscopic and microscopic particle velocities. To establish values of **W** and ρ it is necessary to average the quantities over a region with dimensions $\gtrsim\Delta$. While dephasing at a distance Δ was insignificant, this averaging did not change the form of the formulas for **W** and ρ. If over the course of a large time $t_1 \sim \Omega/V(d\Omega/dr)$ dephasing becomes significant, this averaging indicates that a portion of the energy of the initial perturbation has transformed to a chaotic form. Interactions not considered explicitly lead to absorption of this portion of the energy which has become chaotic (system heating). We have

$$\langle \mathbf{W} \rangle = \frac{1}{\delta V} \int\limits_{\delta V} \mathbf{W}\, dV \tag{4.7}$$

where $\delta V \sim \Delta^3$. This integral can be evaluated if we introduce a characteristic function of form

$$f(r') = \frac{1}{\sqrt{\pi}\,\Delta} \exp\left\{-\frac{(r'-r)^2}{\Delta^2}\right\} \tag{4.8}$$

and carry the integration limits to infinity. Performing the integration, we have

$$\langle W_r \rangle = V(r)\, e^{-t^2/\tau^2} \cos\Omega(r)\, t \tag{4.9}$$

where $\tau \sim (\Delta(d\Omega/dr))^{-1}$. The density ρ_1, determined from the continuity equation also decays exponentially with time. As follows from Eq. (4.4), $\tau \ll t_1 \sim \Omega\,(V(d\Omega/dr))^{-1}$, i.e., oscillations damp out before the linear approximation becomes invalid. It should be noted that the system is linearly unstable relative to perturbations disrupting Eq. (4.4)(over a time $t_1 \sim \Omega(V(d\Omega/dr))^{-1}$ the density ρ_1 attains values of the order of ρ_0).

In the case of arbitrary perturbations not having central symmetry ($l \neq 0$) it is also necessary to perform supplementary averaging. Calculations with the characteristic function (4.8) give

$$\langle W_r(r, \theta, \varphi, t) \rangle = \Big\{ \sum_{s=-l}^{l} g_s(r)\, e^{-t^2/\tau_s^2} \cos[\omega_B(r, s)\, t + \psi_s(r)]$$
$$+ \sum_{i=1}^{l+1} f_i(r)\, e^{-t^2/\tau_i'^2} \cos[\omega_i(r)\, t + \varphi_i(r)] + G(r, t)\Big\} T^l_{m0}(\pi/2 - \varphi, \theta, 0) \tag{4.10}$$
$$\tau_s \sim \left(\Delta \frac{d\omega_B(r, s)}{dr}\right)^{-1}, \quad \tau_i' \sim \left(\Delta \frac{d\omega_i(r)}{dr}\right)^{-1}$$

where the function $G(r, t)$ at large times decays as t^{-1}. The density $\rho_1(r, t)$ decays with time with the same damping index.

It has been shown that an aggregate with a mass distribution satisfying the condition $d\Omega^2/dr < 0$ is stable relative to arbitrary perturbations. Oscillations which develop damp out with time.

The authors thank Ya. B. Zel'dovich for his stimulating discussions of the study.

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